An exactly soluble model with tunable p-wave paired fermion ground states

Yue Yu[1] and Ziqiang Wang[2]

[1]*Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100080, China*
[2]*Department of Physics, Boston College, Chestnut Hill, MA 02467*

(Dated: January 16, 2009)

Motivated by the work of Kitaev, we construct an exactly soluble spin-$\frac{1}{2}$ model on honeycomb lattice whose ground states are identical to $\Delta_{1x}p_x + \Delta_{1y}p_y + i(\Delta_{2x}p_x + \Delta_{2y}p_y)$-wave paired fermions on square lattice, with tunable paring order parameters. We derive a universal phase diagram for this general p-wave theory which contains a gapped A phase and a topologically non-trivial B phase. We show that the gapless condition in the B phase is governed by a generalized inversion (G-inversion) symmetry under $p_x \leftrightarrow \frac{\Delta_{1y}}{\Delta_{1x}}p_y$. The G-inversion symmetric gapless B phase near the phase boundaries is described by 1+1-dimensional gapless Majorana fermions in the asymptotic long wave length limit, i.e. the $c = 1/2$ conformal field theory. The gapped B phase has G-inversion symmetry breaking and is the weak pairing phase described by the Moore-Read Pfaffian. We show that in the gapped B phase, vortex pair excitations are separated from the ground state by a finite energy gap.

arXiv:0708.0631v3 [cond-mat.str-el] 16 Jan 2009

I. INTRODUCTION

The low energy excitations of a topologically non-trivial phase have remarkable properties. A well-known example is the quasihole excitation of the Laughlin state in the fractional quantum Hall effect (FQHE), which carries fractional charge and anyon statistics. The most intriguing possibility of a topological phase of matter is the nonabelian FQHE proposed by Moore and Read[1] for even-denominator filling factors, e.g. $\nu = \frac{5}{2}$[2]. The quasiparticle excitations, vortices in the Moore-Read Pfaffian wave function, have nonabelian statistics[1] which plays a fundamental role in topological quantum computation[3,4]. The key ingredient in the Moore-Read Pfaffian state is that the topologically nontrivial part of the wave function is asymptomatically the same as the pair wave function in a $p_x + ip_y$-wave fermion paired state[6] in the weak pairing phase[7]. The existence of the exotic non-abelian statistics is thus likely a generic property of the more tangible time-reversal symmetry (T-symmetry) breaking p-wave pairing states.

Recently, Kitaev constructed a spin-$\frac{1}{2}$ model with link-dependent Ising couplings on the honeycomb lattice[4]. Kitaev showed that the model is equivalent to a bilinear Majorana fermion model and is thus exactly soluble. A topological non-trivial gapless phase (the B phase) was discovered.(A Jordan-Wigner transformation to a model with two-Majorana fermions for this model has been proposed in[5]). In the presence of a T-symmetry breaking term, the B phase becomes gapped and exhibits vortex excitations obeying nonabelian statistics. The model also has a topologically trivial, gapped A phase. The two phases are separated by a topological phase transition via a gapless critical state. These properties strongly resemble the weak and strong pairing phases and the critical state in the $p_x + ip_y$-wave paired states of spinless fermions[7].

The interconnections among the Kitaev model, the p-wave paired fermions, and the Moore-Read Pfaffian and its excitations have not been well understood previously. In particular, it is important to understand the universal properties among these systems, analogous to finding the universality class in statistical mechanics models. In this paper, we show that the Kitaev model is a special case of a broader class of two-dimensional spin-$\frac{1}{2}$ models whose ground states are equivalent to general paired fermion states in the p-wave channel. Indeed, the vortex-free Kitaev Hamiltonian maps to an exact BCS fermion pairing model with $i(p_x + p_y)$-wave attractions on a square lattice[9]. Our generalized model includes both the $p_x + ip_y$ wave paired states and the original Kitaev model as special limits. It is an exactly soluble model with minimal three and four-spin interactions. We show that the vortex-free ground states of this model are described by $\Delta_{1x}p_x + \Delta_{1y}p_y + i(\Delta_{2x}p_x + \Delta_{2y}p_y)$-wave paired fermion states with tunable pairing order parameters Δ_{ab} on a square lattice. We find that the structure of the phase diagram is determined by the geometry of the underlying Fermi surface. It contains both topologically trivial (A) and nontrivial (B) phases. The A phase is always gapped and corresponds to the strong pairing phase. The B phase can be either gapped or gapless even if T-symmetry is broken. We find that gapless excitations in the B phase is protected by a generalized inversion (G-inversion) symmetry under $p_x \leftrightarrow \frac{\Delta_{1y}}{\Delta_{1x}}p_y$ and the emergence of a gapped B phase is thus tied to G-inversion symmetry breaking. For instance, the $p_x + ip_y$ wave paired state is gapped while $p_y + ip_y$-wave paired state is gapless although they both break the T-symmetry. The critical states of the A-B phase transition remains gapless whether or not T- and G-inversion symmetries are broken, indicative of its topological nature. Indeed, if all Δ_{ab} are tuned to zero, the topological A-B phase transition is from a band insulator to a free Fermi gas. The Fermi surface shrinks to a point zero at criticality.

We show that the gapped B phase is a weak pairing state while the G-inversion symmetric ground states are extended. The gapless phase was not well-understood before. We show that the effective theory near the phase boundary corresponds to 1+1-dimensional massless Majorana fermions in the long wave length limit, i.e., a $c = 1/2$ conformal field theory or the 2-dimensional Ising model. The vortex excitations are important in the family of Kitaev models since the vortex excitations may obey anyon statistics[4]. The vortex excitation energies have been numerically estimated in the A phase[10] and

the B phase[11]. We study the vortex excitations in the gapped B phase in the continuum limit and show that the vortex pair excitations cost a finite energy. This is consistent with the results of numerical calculations[11] and suggests that vortex excitations may have well-defined statistics.

II. THE GENERALIZATION OF KITAEV MODEL

We extend the Kitaev model on the honeycomb lattice by introducing minimal three- and four-spin terms in the Hamiltonian,

$$
\begin{aligned}
H &= -J_x \sum_{x-links} \sigma_i^x \sigma_j^x - J_y \sum_{y-links} \sigma_i^y \sigma_j^y - J_z \sum_{z-links} \sigma_i^z \sigma_j^z \\
&- \kappa_x \sum_b \sigma_b^z \sigma_{b+e_z}^y \sigma_{b+e_z+e_x}^x \\
&- \kappa_x \sum_w \sigma_w^x \sigma_{w+e_x}^y \sigma_{w+e_x+e_z}^z \\
&- \kappa_y \sum_b \sigma_b^z \sigma_{b+e_z}^x \sigma_{b+e_z+e_y}^y \\
&- \kappa_y \sum_w \sigma_w^y \sigma_{w+e_y}^x \sigma_{w+e_y+e_z}^z \\
&- \lambda_x \sum_b \sigma_b^z \sigma_{b+e_z}^y \sigma_{b+e_z+e_x}^y \sigma_{b+e_z+e_x+e_z}^z \\
&- \lambda_y \sum_b \sigma_b^z \sigma_{b+e_z}^x \sigma_{b+e_z+e_y}^x \sigma_{b+e_z+e_y+e_z}^z ,
\end{aligned} \tag{1}
$$

where $\sigma^{x,y,z}$ are Pauli matrices, x-,y-,z-links are shown in Fig. 1(upper panel), $'w'$ and $'b'$ label the white and black sites of lattice, and e_x, e_y, e_z are the positive unit vectors, which are defined as, e.g., $e_{12} = e_z, e_{23} = e_x, e_{61} = e_y$. $J_{x,y,z}$, $\kappa_{x,y}$ and $\lambda_{x,y}$ are tunable real parameters. The original Kitaev model has $\kappa_\alpha = \lambda_\alpha = 0$, $\alpha = x, y$. Adding a T-symmetry breaking external magnetic field corresponds to $\kappa_x = \kappa_y = \kappa \neq 0$ and a κ_z-term[4,5]. It is important to note that the generalized Hamiltonian maintains the Z_2 gauge symmetry acted by a group element, e.g.,

$$W_P = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

with $[H, W_P] = 0$. In fact, one can construct Z_2 gauge invariant spin models with higher multi-spin terms, e.g., $\sigma_9^z \sigma_{10}^y \sigma_1^y \sigma_2^y \sigma_3^x, \sigma_9^z \sigma_{10}^y \sigma_1^y \sigma_2^y \sigma_3^z \sigma_4^y$ and $\sigma_9^z \sigma_{10}^y \sigma_1^y \sigma_2^y \sigma_3^y \sigma_{16}^z$, and so on. One can also add the 'z'-partners of $\kappa_{x,y}$ and $\lambda_{x,y}$ terms so that the model becomes more symmetric. However, adding these term or not will not affect our result in this paper as we will explain later.

We now write down the Majorana fermion representation of this spin model. Let $b_{x,y,z}$ and c be the four kinds of Majorana fermions with $b_{x,y,z}^2 = 1$ and $c^2 = 1$. The spin operator is given by

$$\hat{\sigma}^a = \frac{i}{2}(b_a c - \frac{1}{2}\epsilon_{abc} b_b b_c).$$

Restricting to the physical Hilbert space, one needs to require[4]

$$D = b_x b_y b_z c = 1$$

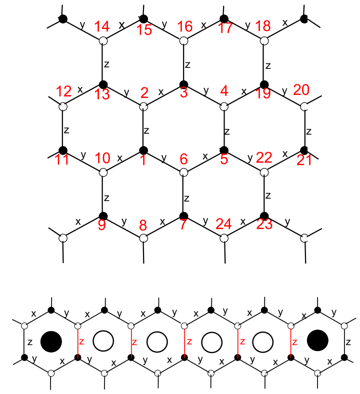

FIG. 1: (Color online) Upper panel: The honeycomb lattices and links . Lower panel: The vortex excitations. The empty circles denote $W_P = 1$ and the filled circles denote vortices with $W_P = -1$.

and thus $\hat{\sigma}^a = i b_a c$. The Hamiltonian now reads

$$
\begin{aligned}
H &= i \sum_a \sum_{a-links} J_a u_{ij}^a c_i c_j - i \sum_b K_{b,b+e_z}^x c_b c_{b+e_z+e_x} \\
&- i \sum_w K_{w+e_x-e_x, w-e_z-e_x}^x c_{w+e_x-e_x} c_{w-e_z-e_x} \\
&- i \sum_b \Lambda_{b,b+2e_z+e_x}^x c_b c_{b+2e_z+e_x} \\
&- i \sum_w \Lambda_{w,w-2e_z-e_x}^x c_w c_{w-2e_z-e_x} \\
&+ y-\text{partners}
\end{aligned} \tag{2}
$$

where $K_{b,b+e_z}^x = \kappa_x u_{b,b+e_z}^z u_{b+e_z+e_x,b+e_z}^x$, $\Lambda_{b,b+2e_z+e_x}^x = \lambda_x u_{b,b+e_z}^z u_{b+e_z,b+e_z+e_x}^x u_{b+e_z+e_x,b+e_z+e_x+e_z}^z$ etc and $u_{ij}^a = i b_i^a b_j^a$ on a-links. It can be shown that the Hamiltonian commutes with u_{ij}^a and thus the eigenvalues of $u_{ij}^a = \pm 1$ because $(u_{ij}^a)^2 = 1$. Since the four spin terms we introduced are related to the hopping between the 'b' and 'w' sites, Lieb's theorem[8] is still applicable. The third spin terms are 'b' to 'b' and 'w' to 'w' and Lieb's theorem is not directly applicable. However, according to Kitaev[4], one can still take $u_{bw}^a = -u_{wb}^a = 1$. The Hamiltonian for the ground state free of the Z_2 vortices ($W_P = 1$ for all P) is given by

$$
\begin{aligned}
H_0 &= i\tilde{J}_x \sum_s (c_{s,b} c_{s-e_x,w} - c_{s,w} c_{s-e_x,b}) \\
&+ i\tilde{\lambda}_x \sum_s (c_{s,b} c_{s-e_x,w} + c_{s,w} c_{s-e_x,b}) \\
&+ i\frac{\kappa^x}{2} \sum_s (c_{s,b} c_{s+e_x,b} + c_{s,w} c_{s-e_x,w}) \\
&+ y \text{ partners} + iJ_z \sum_s c_{s,b} c_{s,w}
\end{aligned} \tag{3}
$$

where s represents the position of a z-link, $\tilde{\lambda}_\alpha = \frac{J_\alpha + \lambda_\alpha}{2}$ and $\tilde{J}_\alpha = \frac{J_\alpha - \lambda_\alpha}{2}$.

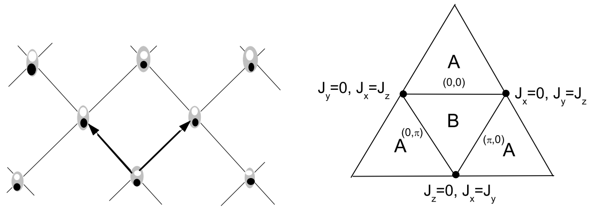

FIG. 2: Left panel: The effective square lattice. Right panel: Phase diagram in the space spanned by $(\tilde{J}_x, \tilde{J}_y, J_z)$. This is a (1,1,1) cross section with all J's positive.

III. MAPPING TO A p-WAVE PAIRED STATE

The next step is to map the *Majorana fermion* H_0 to a fermion model. Defining fermions on the z-links by[9]

$$d_s = (c_{s,b} + ic_{s,w})/2, \quad d_s^\dagger = (c_{s,b} - ic_{s,w})/2,$$

H_0 becomes

$$\begin{aligned} H_0 &= J_z \sum_s (d_s^\dagger d_s - 1/2) + \tilde{J}_x (d_s^\dagger d_{s+e_x} - d_s d_{s+e_x}^\dagger) \\ &+ \tilde{\lambda}_x \sum_s (d_{s+e_x}^\dagger d_s^\dagger - d_{s+e_x} d_s) \\ &+ i\kappa_x \sum_s (d_s d_{s+e_x} + d_s^\dagger d_{s+e_x}^\dagger) + y \text{ partners.} \end{aligned} \tag{4}$$

This is a quadratic model of spinless fermions d_s on the square lattice (Fig. 2, left panel) with general p-wave pairing. If we include the 'z' partner of the three and four spin terms in eq. (1), we have additional corresponding terms in eq. (4) which are the next nearest neighbor terms in the square lattice. These terms will not qualitatively affect our result. Returning to Majorana fermions, a link fermion is a superposition of two Majorana fermions in a link. Therefore, the paring of the link fermions reflects the 'pairing' of Majorana fermions.

After a Fourier transformation, eq. (4) becomes

$$\begin{aligned} H_0 &= \sum_{\mathbf{p}} \xi_{\mathbf{p}} d_{\mathbf{p}}^\dagger d_{\mathbf{p}} + \frac{\Delta_{1,\mathbf{p}}}{2} (d_{\mathbf{p}}^\dagger d_{-\mathbf{p}}^\dagger + d_{\mathbf{p}} d_{-\mathbf{p}}) \\ &+ i\frac{\Delta_{2,\mathbf{p}}}{2} (d_{\mathbf{p}}^\dagger d_{-\mathbf{p}}^\dagger - d_{\mathbf{p}} d_{-\mathbf{p}}) \end{aligned} \tag{5}$$

where the dispersion and the pairing functions are

$$\begin{aligned} &\xi_{\mathbf{p}} = J_z - \tilde{J}_x \cos p_x - \tilde{J}_y \cos p_y, \\ &\Delta_{a,\mathbf{p}} = \Delta_{ax} \sin p_x + \Delta_{ay} \sin p_y, \quad a = 1, 2 \end{aligned}$$

with $\Delta_{1,x(y)} = \kappa_{x(y)}$ and $\Delta_{2,x(y)} = \tilde{\lambda}_{x(y)}$. We have thus shown that the ground state of the extended Kitaev model in Eq. (1) are equivalent to general p-wave paired fermion states. The quasiparticle excitations are governed by the BdG equations

$$E_{\mathbf{p}} u_{\mathbf{p}} = \xi_{\mathbf{p}} u_{\mathbf{p}} - \Delta_{\mathbf{p}}^* v_{\mathbf{p}}, \;\; E_{\mathbf{p}} v_{\mathbf{p}} = -\xi_{\mathbf{p}} v_{\mathbf{p}} - \Delta_{\mathbf{p}} u_{\mathbf{p}} \tag{6}$$

where $E_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 + (\Delta_{1,\mathbf{p}})^2 + (\Delta_{2,\mathbf{p}})^2}$ is the dispersion, $\Delta_{\mathbf{p}} = \Delta_{1,\mathbf{p}} + i\Delta_{2,\mathbf{p}}$, and (u_p, v_p) are the coherence factors with $|u_{\mathbf{p}}|^2 = \frac{1}{2}(1 + \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}), |v_{\mathbf{p}}|^2 = \frac{1}{2}(1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}})$ and $v_{\mathbf{p}}/u_{\mathbf{p}} = -(E_{\mathbf{p}} - \xi_{\mathbf{p}})/\Delta_{\mathbf{p}}^*$.

IV. PHASE DIAGRAM IN TERMS OF THE p-WAVE STATES

We now turn to the properties of this p-wave paired state of link fermions. It is instructive to consider the free fermion dispersion. The condition $\xi_p = 0$ defines a topological transition between a band insulator and a metal with a Fermi surface in the absence of pairing. The solution is given by $|\cos p_x^*| = |\cos p_y^*| = 1$, $\mathbf{p}^* = (0,0), (0,\pm\pi), (\pm\pi,0), (\pm\pi,\pm\pi)$, where $J_z \pm \tilde{J}_x \pm \tilde{J}_y = 0$. Without loss of the generality, one considers only $\tilde{J}_{x,y} > 0$ and $J_z > 0$. Then, $\xi_{p^*} = 0$ corresponds to the inner triangle of the (1,1,1)-cross section in $(\tilde{J}_x, \tilde{J}_y, J_z)$ space (see Fig. 2 (right panel)). Notice that the p-wave pairing gap functions $\Delta_{a,\mathbf{p}}$ vanish at $\mathbf{p}^*$ and therefore, this triangle is the gapless critical boundary separating the A and B phases. Outside the triangle, $\xi_p > 0$. Thus the A phase is gapped. In the limit $\mathbf{p} \to \mathbf{p}^*$, the pair correlation $g_p \equiv v_p/u_p$ is analytic near $\mathbf{p}^*$, implying tightly bound pairs in positional space and hence the A-phase as the strong-pairing phase (See below for detailed discussions)[7]. The global structure of the phase diagram is invariant in the generalized $(\tilde{J}_x, \tilde{J}_y, J_z)$ space because our minimal three- and four-spin extension does not change the topology of the underlying Fermi surface.

The nature of the B phase is much more intriguing. Inside the triangle, ξ_p, $\Delta_{1,\mathbf{p}}$ and $\Delta_{2,\mathbf{p}}$ can be zero individually. The gapless condition ($E_p = 0$) requires all three to be zero at a common $\mathbf{p}^*$. This can only be achieved if (i) one of the $\Delta_{a,\mathbf{p}} = 0$ or (ii) $\Delta_{1,\mathbf{p}} \propto \Delta_{2,\mathbf{p}}$. If either (i) or (ii) is true, $\xi_{\mathbf{p}}$ and $\Delta_{\mathbf{p}}$ can vanish simultaneously, i.e. $E_p = 0$ at $\mathbf{p}^*$, and the paired state is gapless. Otherwise, the B phase is gapped. Note that contrary to conventional wisdom, T-symmetry breaking alone does not guarantee a gap opening in the B phase. The symmetry reason behind the gapless condition of the B phase becomes clear in the continuum limit where $E_p = 0$ implies that the vortex-free Hamiltonian must be invariant, up to a constant, under the transformation $p_x \leftrightarrow \eta p_y$ and $\tilde{J}_x \leftrightarrow \eta^{-2}\tilde{J}_y$ with $\eta = \frac{\Delta_{a,y}}{\Delta_{a,x}}$ with $a = 1$ or 2 and for nonzero Δ. We refer to this as a *generalized inversion (G-inversion) symmetry* since it reduces to the usual mirror reflection when $\eta = 1$. This (projective) symmetry protects the gapless nature of fermionic excitations and may be associated with the underlying quantum order[12]. Kitaev's original model has $\Delta_{1,i} = 0$, and is thus G-inversion invariant and gapless. The magnetic field perturbation[4] breaks this G-inversion symmetry and the fermionic excitation becomes gapped. A special case with G-inversion symmetry breaking is $\Delta_{\mathbf{p}} \propto \sin p_x + i \sin p_y$, i.e. the $p_x + ip_y$-wave paired state discussed by Read and Green in the continuum limit[7].

The N-fermion ground state wave function in the general p-wave paired state can

be written down as a Pfaffian for N even: $\Psi(\mathbf{r}_1,...,\mathbf{r}_1) = 1/(2^{N/2}(N/2)!)\sum_P \mathrm{sgn}P\prod_{i=1}^{N/2} g(\mathbf{r}_{P_{2i-1}} - \mathbf{r}_{P_{2i}})$ where $g(r)$ is the "pair correlation", i.e. the Fourier transform of $g_{\mathbf{p}} = v_{\mathbf{p}}/u_{\mathbf{p}}$ in the BCS wave function $|\Omega\rangle = \prod_{\mathbf{p}}|u_{\mathbf{p}}|^{1/2}\exp\left(\frac{1}{2}\sum_{\mathbf{p}} g_{\mathbf{p}} d^\dagger_{\mathbf{p}} d^\dagger_{-\mathbf{p}}\right)|0\rangle$. The wave function exhibits very different behaviors in the long wave length limit in different parameters[13]. In the A phase, $\xi_p > 0$ as $\mathbf{p} \to 0$, thus $g_{\mathbf{p}} \propto \Delta_{\mathbf{p}}$. The analyticity of $g_{\mathbf{p}}$ leads to $g(\mathbf{r}) \propto e^{-\mu r}$ as in the strong pairing phase of a pure $p_x + ip_y$ state[7]. In the gapped B phase with G-inversion symmetry breaking, $\xi_p < 0$ as $\mathbf{p} \to 0$. Defining $p'_i = \Delta_{ai}p_i$ with $a = 1, 2$ and $i = x, y$, it follows that $g_{\mathbf{p}} \propto \frac{1}{p'_x + ip'_y}$, leading to $g(\mathbf{r}) = \frac{1}{x'_1 + ix'_2}$ with $x'_a = \Delta^{-1}_{ai}x_i$ and thus a weak-pairing phase. Identifying $z' = x'_1 + ix'_2$, we see that the ground state of the gapped B phase corresponds exactly to the Moore-Read Pfaffian. It is easy to show that, u and v^* obey the same BdG equation in this general p-wave paired state, such that the anti-particle of the quasiparticle $\psi = (u, v)$ is itself, i.e., it is Majorana fermion obeying Dirac equations in 2+1-dimensions[7].

We now discuss the nature of the gapless B phase in the general model with G-inversion symmetry. In this case, $E_{\mathbf{p}} = 0$ at $\mathbf{p} = \pm\mathbf{p}^*$ which are the solutions of $\xi_{\mathbf{p}} = 0$ and, say, $\Delta_{\mathbf{p}} = \Delta_{1,\mathbf{p}} = 0$. At $\mathbf{p}^*$, the fermion dispersions are generally given by 2D Dirac cones. However, by a continuous variation of the parameters, one can realize a dimensional reduction near the phase boundary where the effective theory is in fact a 1+1 dimensional conformal field theory in the long wave length limit. Let us consider parameters that are close to the critical line with $|\sin p^*_a| \ll |\cos p^*_a|$ where $g_{\mathbf{q}} = \mathrm{sgn}[q_x\Delta_{1x}\cos p^*_x + q_y\Delta_{1y}\cos p^*_y] \equiv \mathrm{sgn}(q'_x)$ with $\mathbf{q} = \mathbf{p} - \mathbf{p}^*$. Doing the Fourier transform, we find

$$g(\mathbf{r}) = \int dq'_x dq'_y e^{iq'_x x' + iq'_y y}\mathrm{sgn}(q'_x) = \delta(y')\int dq'_x \frac{q'_x}{|q'_x|}\sin q'_x x' \sim \frac{\delta(y')}{x'}. \quad (7)$$

The $\delta(y)$-function indicates that the pairing in the gapless B phase has a one-dimensional character and the ground state is a one-dimensional Moore-Read Pfaffian. The BdG equations reduce to

$$i\partial_t u = -i\Delta_{1x}(1 + i\eta)\partial_{x'}v, \;\; i\partial_t v = i\Delta_{1x}(1 - i\eta)\partial_{x'}u, \quad (8)$$

with $\eta = \frac{\Delta_{1y}}{\Delta_{1x}}$. Thus, the gapless Bogoliubov quasiparticles are one-dimensional Majorana fermions. The long wave length effective theory for the gapless B phase near the phase boundary is therefore the massless Majorana fermion theory in 1+1-dimensional space-time, i.e. a $c = 1/2$ conformal field theory or equivalently a two-dimensional Ising model.

V. TOPOLOGICAL INVARIANT IN A AND B PHASES

We note that there is no spontaneous breaking of a continuous symmetry associated with the phase transition from A to B phases. Kitaev has shown that the A phase in his model is topologically trivial and has zero spectral Chern number, while the gapped B phase is characterized by the Chern number ± 1[4]. This fact was already discussed by Read and Green in the context of $p_x + ip_y$ paired state. Here we follow Read and Green[7] to study the topological invariant in a general p-wave state.

In continuum limit, $\mathbf{p} = (p_x, p_y)$ lives in an Euclidean space R^2. However, the constraint $|u_p|^2 + |v_p|^2 = 1$ parameterizes a sphere S^2. As $|\mathbf{p}| \to \infty$, $\xi_{\mathbf{p}} \to E_{\mathbf{p}}$ such that $v_{\mathbf{p}} \to 0$. Therefore, we can compactify R^2 into an S^2 by adding ∞ to R^2 where $v_{\mathbf{p}} \to 0$. The sphere $|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 = 1$ can also be parameterized by a pseudospin vector $\mathbf{n}_{\mathbf{p}} = (\Delta_{1,\mathbf{p}}, -\Delta_{2,\mathbf{p}}, \xi_{\mathbf{p}})/E_{\mathbf{p}}$ because $|\mathbf{n}_{\mathbf{p}}| = 1$. $(u_{\mathbf{p}}, v_{\mathbf{p}})$ thus describes a mapping from S^2 ($\mathbf{p} \in R^2$) to S^2 (spinor $|\mathbf{n}_{\mathbf{p}}| = 1$). The winding number of the mapping is a topological invariant.The north pole is $u_{\mathbf{p}} = 1, v_{\mathbf{p}} = 0$ at $|\mathbf{p}| = \infty$ and the south pole is $u_{\mathbf{p}} = 0, v_{\mathbf{p}} = 1$ at $\mathbf{p} = \mathbf{0}$. In the $\mathbf{n}_{\mathbf{p}}$ parametrization, $\mathbf{n}_{\mathbf{0}} = (0, 0, \frac{\xi_{\mathbf{p}}}{E}) = (0, 0, 1)$ at $|\mathbf{p}| = \infty$ and $(0, 0, \frac{\xi_{\mathbf{p}}}{E}) = (0, 0, -1)$ at $\mathbf{p} = 0$, corresponding to either the north pole or south pole.

In the strong pairing phase, we know that $u_p \to 1$ and $v_p \to 0$ as $\mathbf{p} \to 0$ (or equivalently, $\xi_{\mathbf{p}} > 0$). This means that for arbitrary $\mathbf{p}$, $(u_{\mathbf{p}}, v_{\mathbf{p}})$ maps the p-sphere to the upper hemisphere and the winding number is zero. That is, the topological number $\nu = 0$ in the strong pairing phase.

In the weak paring phase, $u_{\mathbf{p}} \to 0$ and $v_{\mathbf{p}} \to 1$ as $\mathbf{p} \to 0$. This means that the winding number is nonzero (at least wrapping once). For our case, the winding number can be directly calculated and is given by

$$\nu = \frac{1}{4\pi}\int dp_x dp_y \mathbf{n}_{\mathbf{p}} \cdot \left(\partial_{p_x}\mathbf{n}_{\mathbf{p}} \times \partial_{p_y}\mathbf{n}_{\mathbf{p}}\right) = 1 \quad (9)$$

Defining $P(\mathbf{p}) = \frac{1}{2}(1 + \mathbf{n}_{\mathbf{p}} \cdot \vec{\sigma})$, which is the Fourier component of the projection operator to the negative spectral space of the Hamiltonian, this winding number can be identified as the spectral Chern number defined by Kitaev[4]

$$\nu = \frac{1}{2\pi i}\int \mathrm{Tr}[P_-(\partial_{p_x}P_-\partial_{p_y}P_- - \partial_{p_y}P_-\partial_{p_x}P_-)]dp_x dp_y \quad (10)$$

where $P_- = I - P$ is a projective operator. This spectral Chern number vanishes in the strong pairing A phase but takes an integer value in the weak paring B phase. Thus, the phase transition from A to B is a topological phase transition.

VI. VORTEX EXCITATIONS

We have discussed the universal behaviors of the ground state. We now turn to discuss the Z_2 vortex excitation in the spin model which corresponds to setting $W_P = -1$ for a given plaquette. The Hamiltonian in the Majorana fermion representation is bilinear and the energies of the vortices can be estimated both in the A phase[10] and the B phase[11]. However, it remained difficult to obtain analytical solutions of the wave functions with two well-separated vortices. We have shown that the ground state sector is equivalent to the $p_x + ip_y$ pairing theory for fermions on the square lattice. Therefore,

the Pfaffian state is the ground state wave function in the continuum limit in the weak pairing phase. Our strategy is to evaluate the energy of the trial wave function containing vortices above the Pfaffian state in the continuum limit. For two well separated half-vortices located at w_1 and w_2 shown in Fig. 1 (lower panel), the Moore-Read trial wave function has been well-studied[1,7] and is given by

$$\Psi(z_1, ...z_N; w_1, w_2) \propto \mathrm{Pf}(g'(z_i, z_j; w_1, w_2)), \quad (11)$$
$$g'(z_1, z_2; w_1, w_2) \propto \frac{(z_1 - w_1)(z_2 - w_2) + (w_1 \leftrightarrow w_2)}{z_1 - z_2}.$$

The second quantized state corresponding to this wave function reads

$$|w_1, w_2\rangle \propto \exp\{\frac{1}{2}\sum_{\mathbf{r}_1,\mathbf{r}_2} g'(z_1, z_2; w_1, w_2) d^\dagger_{\mathbf{r}_1} d^\dagger_{\mathbf{r}_2}\} \quad (12)$$

Performing a Fourier transformation, we have

$$|w_1, w_2\rangle \propto \exp\{\frac{1}{2}\sum_{\mathbf{K},\mathbf{k}} g'_k(\mathbf{K}) d^\dagger_{\mathbf{K}+\mathbf{k}} d^\dagger_{\mathbf{K}-\mathbf{k}}\},$$

where $\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$ and $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$ are the relative and the total momenta of the pairs and $g'_k(\mathbf{K})$ is the Fourier transform of $g'(\mathbf{r}_1, \mathbf{r}_2)$. One can show that,

$$\begin{aligned} g'_k(\mathbf{K}) &\sim (\frac{1}{k} - \frac{A}{w_1 w_2 |\mathbf{k}|^2 k})\delta(\mathbf{K}) \\ &+\frac{1}{k}(\frac{B}{w_1 w_2 |K|^2 \bar{K}^2} - \frac{(w_1+w_2)C}{w_1 w_2 |K|^2 \bar{K}}) \\ &= g'_k \delta(\mathbf{K}) + \frac{1}{k}\tilde{g}(\mathbf{K}) \end{aligned} \quad (13)$$

where A, B and C are positive constants and $\tilde{g}(\mathbf{K})$ is independent of $\mathbf{k}$. Thus,

$$|w_1, w_2\rangle \propto \exp\{\frac{1}{2}\sum_{\mathbf{k}} g'_k d^\dagger_{\mathbf{k}} d^\dagger_{-\mathbf{k}} + \sum_{\mathbf{K},\mathbf{k}} (1/k) g'(\mathbf{K}) d^\dagger_{\mathbf{K}+\mathbf{k}} d^\dagger_{\mathbf{K}-\mathbf{k}}\}$$

Such a vortex pair is shown in Fig. 1 where the red z-links have $u_{bw} = -1$ and all others have $u_{bw} = 1$. The corresponding Hamiltonian can be written as $H = H_0 + \delta H$, where H_0 is the vortex-free Hamiltonian and δH is the vortex part. The latter is expressed as a sum of the pairing and chemical potential terms according to Eq. (5) over the red z-links extending in the ξ-direction (i.e., the direction with $x = y$) between the vortices. It is straightforward to show that δH has the following expectation value in the vortex state,

$$\begin{aligned} &\langle w_1, w_2|\delta H|w_1, w_2\rangle \quad (14) \\ &\propto \sum_{p_\xi, p'_\xi} \frac{i(e^{iw_1(p_\xi+p'_\xi)} - e^{iw_2(p_\xi+p'_\xi)})}{p_\xi + p'_\xi} f(p_\xi, p'_\xi) = 0, \end{aligned}$$

where $f(p_\xi, p'_\xi)$ is an analytical function of $p_\xi + p'_\xi$. *This means that there are no a continuum spectrum above the vortex pairs and then the vortex pairs are also separated from other higher energy excitations.* On the other hand, one can check that since $[H_0, \sum_{\mathbf{K},\mathbf{k}} g'_k(\mathbf{K} \neq 0) d^\dagger_{\mathbf{K}+\mathbf{k}} d^\dagger_{\mathbf{K}-\mathbf{k}}] = 0$, the $\mathbf{K} \neq 0$ sector does not play a nontrivial role in calculating the energy E_v of such a vortex pair. The latter is given by

$$\begin{aligned} E_v &= \langle w_1, w_2|H|w_1, w_2\rangle = \langle w_1, w_2|H_0|w_1, w_2\rangle \\ &= \sum_{\mathbf{k}} E_{\mathbf{k}} |u_{\mathbf{k}} \delta g_{\mathbf{k}}|^2 \langle w_1 w_2|d_{\mathbf{k}} d^\dagger_{\mathbf{k}}|w_1 w_2\rangle \\ &= \sum_{\mathbf{k}} E_{\mathbf{k}} |u_{\mathbf{k}} \delta g_{\mathbf{k}}|^2/(1 + |g'^0_k|^2), \end{aligned} \quad (15)$$

where $g'^0_k = g'_k + \frac{1}{k}(\frac{B}{w_1 w_2 \bar{K}^2} - \frac{(w_1+w_2)C}{w_1 w_2 \bar{K}})|_{\mathbf{K}\to 0}$ and $\delta g_{\mathbf{k}} = g'^0_k - g_{\mathbf{k}}$. Physically, the factor $\langle w_1 w_2|d_{\mathbf{k}} d^\dagger_{\mathbf{k}}|w_1 w_2\rangle = 1 - |g'^0_k|^2/(1 + |g'^0_k|^2) = 1/(1 + |g'^0_k|^2)$ in Eq.(15) is the quasi-hole distribution when the two vortices are located at w_1 and w_2. Thus, E_v indeed corresponds to the energy cost to excite the vortex pair. We have evaluated the vortex pair energy E_v in different limits. First, if w_1 and w_2 were sent to infinity before $\mathbf{K} \to 0$, then $g'^0_k \to g_k$ and we recover the ground state. Second, if $\mathbf{K} \to 0$ while w_1 and w_2 remain finite, then $E_{\mathbf{k}} |u_{\mathbf{k}} \delta g_{\mathbf{k}}|^2/(1 + |g'^0_k|^2) \sim E_{\mathbf{k}} |u_{\mathbf{k}}|^2$, which tends to $|\mathbf{k}|^2$ in both the small and large k limits. The excitation energy is thus high but finite due to the short distance cut-off. (Recall that $\xi_{\mathbf{k}} < 0$ in the B phase). It is independent of w_1 and w_2 and as a result the vortices are deconfined. The third case is when the vortices are far from the origin such that $|Kw_1|$ and $|Kw_2|$ are finite. In the short distance, large k limit, $E_{\mathbf{k}} |u_{\mathbf{k}} \delta g_{\mathbf{k}}|^2/(1 + |g'^0_k|^2) \sim |\mathbf{k}|^{-4}$. On the other hand, in the long wavelength limit with $|\mathbf{k}| \to 0$, $|u_{\mathbf{k}}| \sim |\mathbf{k}|$, $|g'^0_k| \to |\mathbf{k}|^{-3}$ and $E_{\mathbf{k}} \to$ constant such that $E_{\mathbf{k}} |u_{\mathbf{k}} \delta g_{\mathbf{k}}|^2/(1 + |g'^0_k|^2) \sim |\mathbf{k}|^2 \to 0$. As a result, the vortex pair energy E_v is free of infrared divergences and is only weakly dependent on $w_1 - w_2$. Therefore, the vortices are also deconfined. Finally, since E_k increases with the paring parameters κ and λ in Eq.(4), E_v is expected to increase with increasing paring gap parameters. These results are consistent with the finite size numerical calculations of the vortex pair energy in the lattice model[11]. Our analytical results suggest that the vortex pair described by Eq.(11), while costing a high energy in the bulk, corresponds to low energy excitations near the edge of the system.

We note an important difference between this p-wave theory and a conventional p-wave superfluid: Instead of spontaneously breaking the $U(1)$ symmetry in an usual p-wave superfluid, only the discrete Z_2 symmetry is broken and the $U(1)$ symmetry is absent in the present model. The vortices studied here are thus Z_2 vortices instead of $U(1)$ vortices. As a consequence, in the gapped B phase, the vortices are in the deconfinement phase[15] instead of being logarithmically confined in the p-wave superfluid. This fact can be easily seen because $\Delta_{ax(y)}$ are real and there is no $U(1)$ phase factor whose gradient gives rise to a vector field of the vortex. Our analysis of the vortex pair energy also shows this difference.

The finiteness of E_v and the vanishing of $\langle\delta H\rangle = 0$ imply that the vortex excitations are *separated* either from the ground state or other excitations. This is consistent with analysis of Read and Green on the U(1) vortex excitations in the p-wave paired state[7,14]. To determine how close the Moore-Read vortex state is to the exact vortex excitations in this

model requires a numerical calculation of the overlapping between the exact eigenstates and the Moore-Read vortex wave functions. A more important question is the realization of the Read-Moore four-vortex state which has a two-fold degeneracy with the vortices obeying non-abelian statistics[1]. We leave these studies to future works.

VII. CONCLUSIONS

We have constructed an exactly soluble spin model with two-, three- and four-spin couplings on a honeycomb lattice. The ground state sector of this model on the honeycomb lattice is mapped to a p-wave paired state of the link fermions on a square lattice with general pairing parameters. Based on the general p-wave paired states, we analyzed the phase diagram of the system and the properties of topologically different phases. We found that our phase diagram is universal and includes both the Kitaev model and the Pfaffian state in its universality class.

The authors thank the KITPC for warm hospitality where this work was initiated and finalized during the Program on Quantum Phases of Matter. The authors are grateful to Z. Nussinov, N. Read, X. Wan, X.-G. Wen, K. Yang, J. W. Ye, and Y. S. Wu for useful discussions. This work was supported in part by NNSF of China, the Ministry of Science and Technology of China, a fund from CAS and the U.S. DOE Grant No. DE-FG02-99ER45747.

[1] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[2] J. S. Xia, W. Pan, C. L. Vincente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, K. W. West, Phys. Rev. Lett. **93**, 176809 (2004).
[3] A. Kitaev, Ann. Phys. **303**, 2(2003). M. H. Freedman, M. Larsen, Z. H. Wang, Commun. Math. Phys. **227**, 605(2002).
[4] A. Kitaev, Ann. Phys. **321**, 2(2006).
[5] X. Y. Feng, G. M. Zhang, and T. Xiang, Phys. Rev. Lett. **98**, 087204 (2007).
[6] M. Greiter, X.G. Wen and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[7] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[8] E. H. Lieb, Phys. Rev. Lett. **73**, 2158 (1994).
[9] H. D. Chen and J. P. Hu, Phys. Rev. B **76**, 193101 (2007). H. D. Chen and Z. Nussinov, J. Phys. A **41**, 075001 (2008).
[10] K. P. Schmidt, S. Dusuel and J. Vidal, Phys. Rev. Lett. **100**, 057208 (2008).
[11] V. Lahtinen, G. Kells, A. Carollo, T. Stitt, J. Vala and J. K. Pachos, Ann. of Phys. **323**, 2286 (2008).
[12] X. G. Wen and A. Zee, Phys.Rev. B **66**, 235110 (2002).
[13] There are other gapless lines corresponding to $\mathbf{p}^* = (0, \pi)$ and $(\pi, 0)$. The Pfaffian and anti-Pfaffian states may be discussed near these points. For the anti-Pfaffian, see, S. S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Phys. Rev. Lett. **99**, 236807 (2007) ; M. Levin, B. Halperin, and B. Roscnow, Phys. Rev. Lett. **99**, 236806 (2007).
[14] D. H. Lee, G. M. Zhang, and T. Xiang, Phys. Rev. Lett. **99**, 196805 (2007).
[15] T. Senthil and M. P. Fisher, Phys. Rev. B **62**, 7850 (2000).